

THE STEINBERG CURVE

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ABSTRACT. Let E and E' be elliptic curves over \mathbb{C} , with Tate parametrizations $p : \mathbb{C}^* \rightarrow E$, $p' : \mathbb{C}^* \rightarrow E'$. We have the map $p * p' : \mathbb{C}^* \otimes \mathbb{C}^* \rightarrow F^2\mathrm{CH}^2(E \times E')$ sending $u \otimes v$ to the class of the zero cycle $(x, y) - (x, 0) - (0, y) + (0, 0)$, where $x = p(u)$, $y = p'(v)$. We show that, for general $u \in \mathbb{C}^*$, $p * p'(u \otimes (1 - u))$ is not zero in CH^2 . We also show that the cycle $p * p'(u \otimes (1 - u))$ is not detectable by a certain class of cohomology theories, including the cohomology of the analytic motivic complex involving the dilogarithm function defined by S. Bloch in [3]. This is in contrast to its étale version defined by S. Lichtenbaum [8], which contains the Chow group.

1. TATE CURVES AND LINE BUNDLES

For a scheme X over \mathbb{C} , we let X_{an} denote the set of \mathbb{C} -points with the classical topology. We let $\mathcal{O}_{X_{\mathrm{an}}}$ denote the sheaf of holomorphic functions on X_{an} .

We begin by describing a construction of the universal analytic Tate curve over \mathbb{C} . We first form the analytic manifold $\hat{\mathcal{C}}^*$ as the quotient of the disjoint union $\sqcup_{i=-\infty}^{\infty} U_i$, with each $U_i = \mathbb{C}^2$, by the equivalence relation

$$(x, y) \in U_i \setminus \{Y = 0\} \sim \left(\frac{1}{y}, xy^2\right) \in U_{i+1} \setminus \{X = 0\}.$$

The function $\tilde{\pi}(x, y) = xy$ on $\hat{\mathcal{C}}^*$ is globally defined. Letting $D \subset \mathbb{C}$ be the disk $\{|z| < 1\}$, we define $\mathcal{C}^* = \tilde{\pi}^{-1}(D)$, so $\tilde{\pi}$ restricts to the analytic map $\pi : \mathcal{C}^* \rightarrow D$. We let $\tilde{0} : D \rightarrow \mathcal{C}^*$ be the section $z \mapsto (z, 1) \in U_0$.

Let $D^* \subset D$ be the punctured disk $z \neq 0$. Since the map $(x, y) \mapsto (\frac{1}{y}, xy^2)$ is an automorphism of $(\mathbb{C}^*)^2$, the open submanifold $\pi^{-1}(D^*)$ of \mathcal{C}^* is isomorphic to $(\mathbb{C}^*)^2$, and the restriction of the map π is just

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the map $(x, y) \mapsto xy$. Thus, the projection $p_2 : (\mathbb{C}^*)^2 \rightarrow \mathbb{C}^*$ gives an isomorphism of the fiber $\mathcal{C}_t^* := \pi^{-1}(t)$ with \mathbb{C}^* , for $t \in D^*$.

The fiber $\pi^{-1}(0)$, on the other hand, is an infinite union of projective lines. Indeed, define the map $f_i : \mathbb{CP}^1 \rightarrow \mathcal{C}_0^*$ by sending $(a : 1) \in \mathbb{CP}^1 \setminus \infty$ to $(0, a) \in U_i$, and $\infty = (1 : 0)$ to $(0, 0) \in U_{i+1}$, and let $C_i = f_i(\mathbb{CP}^1)$. Then $\pi^{-1}(0) = \cup_{i=-\infty}^{\infty} C_i$, with $\infty \in C_i$ joined with $0 \in C_{i+1}$. Note in particular that the value $\tilde{0}(0)$ of the zero section avoids the singularities of $\pi^{-1}(0)$.

Define the automorphism ϕ of \mathcal{C}^* over D by sending $(x, y) \in U_i$ to $(x, y) \in U_{i-1}$. This gives the action of \mathbb{Z} on \mathcal{C}^* , with n acting by ϕ^n . It is easy to see that this action is free and proper, so the quotient space $\mathcal{E} := \mathcal{C}^*/\mathbb{Z}$ exists as a bundle $\pi : \mathcal{E} \rightarrow D$. The section $\tilde{0} : D \rightarrow \mathcal{C}^*$ induces the section $0 : D \rightarrow \mathcal{E}$.

Take $t \in D^*$. Identifying \mathcal{C}_t^* with \mathbb{C}^* as above, we see that ϕ restricts to the automorphism $z \mapsto tz$. Thus, the fiber $\mathcal{E}_t := \pi^{-1}(t)$ for $t \in D^*$ is the *Tate elliptic curve* $\mathbb{C}^*/t\mathbb{Z}$, with identity $0(t)$. On \mathcal{C}_0^* , however, ϕ is the union of the “identity” isomorphisms $C_i \rightarrow C_{i-1}$. Thus $\phi(\infty \in C_i) = 0 \in C_i$, so the restriction of $\mathcal{C}_0^* \rightarrow \mathcal{E}_0$ to C_0 identifies \mathcal{E}_0 with the nodal curve $\mathbb{CP}^1/0 \sim \infty$. We let $*$ in \mathcal{E}_0 denote the singular point. Then $\tilde{0}(0) \in \mathcal{E}_0 \setminus *$.

The map $(t, w) \in D \times \mathbb{C}^* \mapsto (\frac{t}{w}, w) \in U_0$ gives an isomorphism $\psi : D \times \mathbb{C}^* \rightarrow U_0 \setminus \{Y = 0\}$ over D . The composition

$$D \times \mathbb{C}^* \rightarrow U_0 \setminus \{Y = 0\} \subset \mathcal{C}^* \xrightarrow{q} \mathcal{E}$$

defines the map $p : D \times \mathbb{C}^* \rightarrow \mathcal{E}$ over D , with image $\mathcal{E} \setminus \{*\}$.

Take $u \in \mathbb{C}^*$. We have the local system on \mathcal{E}

$$\mathcal{L}_u := \mathcal{C}^* \times \mathbb{C}/(z, \lambda) \sim (\phi(z), u\lambda) \rightarrow \mathcal{E},$$

and the associated holomorphic line bundle $\mathcal{L}_u^{\text{an}}$ on \mathcal{E} .

Let E_t be the algebraic elliptic curve associated to the analytic variety \mathcal{E}_t , let $L_u(t)$ and $L_u^{\text{an}}(t)$ denote the restriction of \mathcal{L}_u and $\mathcal{L}_u^{\text{an}}$ to \mathcal{E}_t , and let $L_u^{\text{alg}}(t)$ be the algebraic line bundle on E_t corresponding to $L_u^{\text{an}}(t)$ via [13]. The restriction of p to $t \times \mathbb{C}^*$ defines the map $p_t : \mathbb{C}^* \rightarrow E_{t\text{an}}$. For $t \neq 0$, p_t is a covering space of $E_{t\text{an}}$. The map $p_0 : \mathbb{C}^* \rightarrow E_{0\text{an}}$ is the analytic map associated to the algebraic open immersion

$$\mathbb{P}^1 \setminus \{0, \infty\} \xrightarrow{j} \mathbb{P}^1 \rightarrow \mathbb{P}^1/0 \sim \infty = E_0.$$

If E be an elliptic curve over \mathbb{C} , then $E_{\text{an}} \cong \mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice spanned by 1 and some τ in the upper half plane. Taking $t = e^{2\pi i \tau}$ gives the isomorphism $E_{\text{an}} \cong \mathcal{E}_t$, so each elliptic curve over \mathbb{C} occurs as an E_t for some (in fact for infinitely many) $t \in D^*$.

Sending $u \in \mathbb{C}^*$ to the isomorphism class of $L_u^{\text{alg}}(t)$ defines a homomorphism $\tilde{p}_t : \mathbb{C}^* \rightarrow \text{Pic}(E_t)$. We denote the identity $0(t) \in E_t$ simply by 0 if t is given.

Lemma 1.1. *For all $t \in D$, $c_1(L_u^{\text{alg}}(t)) = (p_t(u)) - (0)$.*

Proof. We first handle the case $t \neq 0$. Let $q : \mathbb{C} \rightarrow E := E_t$ be the map $q(z) = p_t(e^{2\pi iz})$, let $\tau \in \mathbb{C}$ be an element with $e^{2\pi i\tau} = t$, and let $\Lambda \subset \mathbb{C}$ be the lattice generated by 1 and τ . The map q identifies E with \mathbb{C}/Λ , and $L_u(t)$ with the local system defined by the homomorphism $\rho : \Lambda \rightarrow \mathbb{C}^*$, $\rho(a + b\tau) = u(b)$.

There is a unique cocycle θ in $Z^1(\Lambda, H^0(\mathbb{C}, \mathcal{O}_{\mathbb{C}_{\text{an}}}^*))$ with $\theta(1) = 1$, $\theta(\tau) = e^{-2\pi iz}$; let L be the corresponding holomorphic line bundle on E . Computing $c_1^{\text{top}}(L) \in H^2(E, \mathbb{Z})$ by using the exponential sequence, we find that $\deg(L) = 1$. By Riemann-Roch, we have $H^0(E, L) = \mathbb{C}$; let $\Theta(z)$ be the corresponding global holomorphic function on \mathbb{C} , i.e.,

$$\Theta(z+1) = \Theta(z), \quad \Theta(z+\tau) = e^{-2\pi iz}\Theta(z),$$

and the divisor of Θ on E is (x) , with $L \cong \mathcal{O}_E(x)$.

Take $v, w \in \mathbb{C}$ with $u = e^{2\pi iv}$ and $q(w) = x$. Let $f(z) = \frac{\Theta(z+w-v)}{\Theta(z+w)}$. Then

$$f(z+1) = f(z), \quad f(z+\tau) = uf(z),$$

and $\text{Div}(f) = (p(u)) - (0)$. Thus, multiplication by f defines an isomorphism

$$\times f : \mathcal{O}_{E_{\text{an}}}((p(u)) - (0)) \rightarrow L_u^{\text{an}}.$$

The proof for $E_0 = \mathbb{P}^1/0 \sim \infty$ is essentially the same, where we replace $\frac{\Theta(z+w-v)}{\Theta(z+w)}$ with the rational function $\frac{X-u}{X-1}$. \square

Thus, the image of \tilde{p}_t in $\text{Pic}(E_t)$ is $\text{Pic}^0(E)$. After identifying the smooth locus of E_t^0 of E_t with $\text{Pic}^0(E_t)$ by sending $x \in E_t^0$ to the class of the invertible sheaf $\mathcal{O}_{E_t}((x) - (0))$, we have $\tilde{p}_t = p_t$.

2. THE ALBANESE KERNEL AND THE STEINBERG RELATION

Let X be a smooth projective variety. We let $\text{CH}_0(X)$ denote the group of zero cycles on X , modulo rational equivalence, $F^1\text{CH}_0(X)$ the subgroup of cycles of degree zero, and $F^2\text{CH}_0(X)$ the kernel of the Albanese map $\alpha_X : F^1\text{CH}_0(X) \rightarrow \text{Alb}(X)$. The choice of a point $0 \in X$ gives a splitting to the inclusion $F^1\text{CH}_0(X) \rightarrow \text{CH}_0(X)$.

Let E, E' be smooth elliptic curves. As $\text{Alb}(E \times E') = E \times E'$, the inclusion $F^2\text{CH}_0(E \times E') \rightarrow F^1\text{CH}_0(E \times E')$ is split by sending $(x, y) - (0, 0)$ to $(x, y) - (x, 0) - (0, y) + (0, 0)$. Thus $F^2\text{CH}_0(E \times E')$

is generated by zero-cycles of the form $(x, y) - (x, 0) - (0, y) + (0, 0)$. Choosing an isomorphism $E \cong E_t$, $E' \cong E_{t'}$, we have the covering spaces $p : \mathbb{C}^* \rightarrow E_{\text{an}}$, $p' : \mathbb{C}^* \rightarrow E'_{\text{an}}$, and the map

$$(2.1) \quad \begin{aligned} p * p' : \mathbb{C}^* \otimes \mathbb{C}^* &\rightarrow F^2\text{CH}_0(E \times E') \\ u \otimes v &\mapsto p(u) * p'(v) := \\ &(p(u), p'(v)) - (p(u), 0) - (0, p'(v)) + (0, 0). \end{aligned}$$

By the theorem of the cube [10], the map $p * p'$ is a group homomorphism, and thus is surjective.

In case one or both of E , E' is the singular curve E_0 , we will need to use the theory of zero-cycles mod rational equivalence defined in [6]. If X is a reduced, quasi-projective variety over a field k with singular locus X_{sing} , the group $\text{CH}_0(X)$ (denoted $\text{CH}_0(X, X_{\text{sing}})$ in [6]) is defined as the quotient of the free abelian group on the regular closed points of X , modulo the subgroup generated by zero-cycles of the form $\text{Div} f$, where f is a rational function on a dimension one closed subscheme D of X such that

1. No irreducible component of D is contained in X_{sing} .
2. In a neighborhood of each point of $D \cap X_{\text{sing}}$, the subscheme D is a complete intersection.
3. f is in the subgroup $\mathcal{O}_{D, D \cap X_{\text{sing}}}^*$ of $k(D)^*$.

It follows in particular from these conditions that $\text{Div} f$ is a sum of regular points of X .

For X a reduced curve, sending a regular closed point $x \in X$ to the invertible sheaf $\mathcal{O}_X(x)$ extends to give an isomorphism $\text{CH}_0(X) \cong \text{Pic}(X)$.

We extend the definition of $F^i\text{CH}_0$ to $E \times E'$ with either $E = E_0$ or $E' = E_0$ or $E = E' = E_0$, by defining $F^1\text{CH}_0(E \times E')$ as the subgroup of $\text{CH}_0(E \times E')$ generated by the differences $[x] - [y]$, and $F^2\text{CH}_0(E \times E')$ the subgroup generated by expressions $[(x, y)] - [(x, 0)] - [(0, y)] + [(0, 0)]$, where x is a smooth point of E and y a smooth point of E' . The surjection $p * p' : \mathbb{C}^* \otimes \mathbb{C}^* \rightarrow F^2\text{CH}_0(E \times E')$ is then defined by the same formula as (2.1).

Proposition 2.1 (The Steinberg relation). *Take $E = E' = E_0$. Then $p(u) * p(1 - u) = 0$ in $\text{CH}_0(E_0 \times E_0)$ for all $u \in \mathbb{C} \setminus \{0, 1\}$.*

Proof. Let X be a quasi-projective surface over a field k . By [7], there is an isomorphism $\phi : H^2(X, \mathcal{K}_2) \rightarrow \text{CH}_0(X)$. The product $\mathcal{O}_X^* \otimes \mathcal{O}_X^* \rightarrow \mathcal{K}_2$ gives the cup product

$$H^1(X, \mathcal{O}_X^*) \otimes H^1(X, \mathcal{O}_X^*) \xrightarrow{\cup} H^2(X, \mathcal{K}_2).$$

In addition, let D, D' be Cartier divisors which intersect properly on X , and suppose that $\text{supp } D \cap \text{supp } D' \cap X_{\text{sing}} = \emptyset$. Then

$$(2.2) \quad \phi(\mathcal{O}_X(D) \cup \mathcal{O}_X(D')) = [D \cdot D'],$$

where \cdot is the intersection product and $[-]$ denotes the class in CH_0 .

Since $L_u^{\text{alg}} = \mathcal{O}_{E_0}(p(u) - 0)$, (2.2) implies

$$p(u) * p(1-u) = \rho(p_1^* L_u^{\text{alg}} \cup p_2^* L_{1-u}^{\text{alg}}),$$

so it suffices to show that $p_1^* L_u^{\text{alg}} \cup p_2^* L_{1-u}^{\text{alg}} = 0$ in $H^2(E_0 \times E_0, \mathcal{K}_2)$.

Write X for $E_0 \times E_0$. Let $\bar{\mathcal{K}}_2$ be the image of \mathcal{K}_2 in the constant sheaf $K_2(\mathbb{C}(X))$. By Gersten's conjecture, the surjection $\pi : \mathcal{K}_2 \rightarrow \bar{\mathcal{K}}_2$ is an isomorphism at each regular point of X , hence π induces an isomorphism on H^2 .

Let $q : \mathbb{P}^1 \rightarrow E_0$ be the normalization, giving the normalization $q \times q : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow X$. Let $i : * \rightarrow E_0$ be the inclusion of the singular point. We have the exact sequence of sheaves on E_0

$$(2.3) \quad q_* \mathcal{K}_1 \xrightarrow{\beta} i_* K_1(\mathbb{C}) \rightarrow 0$$

and the exact sequence of sheaves on X :

$$(2.4) \quad (q \times q)_* \mathcal{K}_2 \xrightarrow{\alpha} (i \times q)_* \mathcal{K}_2 \oplus (q \times i)_* \mathcal{K}_2 \rightarrow (i \times i)_* K_2(\mathbb{C}) \rightarrow 0,$$

with augmentations $\epsilon_1 : \mathcal{K}_1 \rightarrow (2.3)$, $\epsilon_2 : \bar{\mathcal{K}}_1 \rightarrow (2.4)$. The various cup products in K -theory give the map of complexes

$$(2.5) \quad p_1^*(2.3) \otimes p_2^*(2.3) \rightarrow (2.4)$$

over the cup product

$$(2.6) \quad p_1^* \mathcal{K}_1 \otimes p_2^* \mathcal{K}_1 \rightarrow \bar{\mathcal{K}}_2.$$

The augmentation $\epsilon_1 : \mathcal{K}_1 \rightarrow \ker \beta$ is an isomorphism. The augmentation $\epsilon_2 : \bar{\mathcal{K}}_2 \rightarrow \ker \alpha$ is an injection, and the cokernel is supported on $* \times *$, so $\epsilon_2 : \bar{\mathcal{K}}_2 \rightarrow \ker \alpha$ induces an isomorphism on H^2 . Thus, the complexes (2.3) and (2.4) give rise to maps

$$\delta_2 : K_2(\mathbb{C}) \rightarrow H^2(X, \ker \alpha) = H^2(X, \bar{\mathcal{K}}_2) = H^2(X, \mathcal{K}_2)$$

$$\delta_1 : \mathbb{C}^* = K_1(\mathbb{C}) \rightarrow H^1(E_0, \mathcal{K}_1).$$

The compatibility of (2.5) with (2.6) yields the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{C}^* \otimes \mathbb{C}^* & \xrightarrow{\cup} & K_2(\mathbb{C}) \\ \delta_1 \otimes \delta_1 \downarrow & & \downarrow \delta_2 \\ H^1(E_0, \mathcal{K}_1) \otimes H^1(E_0, \mathcal{K}_1) & \xrightarrow{p_1^* \cup p_2^*} & H^2(X, \mathcal{K}_2). \end{array}$$

Since $L_v^{\text{alg}} = \delta_1(v)$ for each $v \in \mathbb{C}^*$, we have

$$p_1^* L_u^{\text{alg}} \cup p_2^* L_{1-u}^{\text{alg}} = \delta_2(\{u, 1-u\}) = 0.$$

□

The main point of this section is that the Steinberg relation is *not* satisfied in $\text{CH}_0(E \times E')$ except in the case $E = E' = E_0$. We first require the following lemma:

Lemma 2.2. *Let $s : \mathbb{C} \setminus \{0, 1\} \rightarrow E \times E'$ be the analytic map $s(u) = (p(u), p'(1-u))$. Then $s(\mathbb{C} \setminus \{0, 1\})$ is not contained in any algebraic curve on $E \times E'$, except in case $E = E' = E_0$.*

Proof. We first consider the case in which both E and E' are smooth elliptic curves, $E = E_t$, $E' = E_{t'}$, where t and t' are in \mathbb{C}^* and $|t| < 1$, $|t'| < 1$. We have the maps

$$p : \mathbb{C}^* \rightarrow E, \quad p' : \mathbb{C}^* \rightarrow E',$$

which are group homomorphisms with $\ker p = t^{\mathbb{Z}}$, $\ker p' = t'^{\mathbb{Z}}$.

Suppose that $s(\mathbb{C}^*)$ is contained in an algebraic curve $D \subset E \times E'$. For each $x \in E$, $(x \times E') \cap D$ is a finite set (possibly empty), hence, for each $u \in \mathbb{C} \setminus \{0, 1\}$, the set of points of $\mathbb{C}^* \times \mathbb{C}^*$ of the form $(t^n u, 1 - t^n u)$ has finite image in $E \times E'$. Thus, for each u , there are integers n, m and p , depending on u , such that $n \neq m$ and

$$(2.7) \quad 1 - t^m u = t'^p (1 - t^n u).$$

Since there are uncountably many u , there is a single choice of n, m and p for which (2.7) holds for uncountably many u . But then

$$(2.8) \quad (t'^p t^n - t^m)u = 1 - t'^p.$$

If $t'^p t^n - t^m = 0$, then $|t'| = 1$, contradicting the condition $|t'| < 1$. If $t'^p t^n - t^m \neq 0$, then we can solve (2.8) for u , so (2.7) only holds for this single u , a contradiction.

If say $E' = E_0$, then $p' : \mathbb{C}^* \rightarrow E'$ is injective, and we have the infinite set of points $p'(1 - t^n u)$ in the image of s , all lying over the single point $p(u)$. □

Theorem 2.3. *Let $E = E_t$, $E' = E_{t'}$, with at least one of E, E' non-singular. Then, for all u outside a countable subset of $\mathbb{C} \setminus \{0, 1\}$, $p(u) * p'(1-u)$ is not a torsion element in $F^2 \text{CH}_0(E \times E')$.*

Proof. We first give the proof in case E and E' are both non-singular. For a quasi-projective \mathbb{C} -scheme X , we let $S^n X$ denote the n th symmetric power of X . For X smooth, we have the map

$$\rho_n : S^n X(\mathbb{C}) \times S^n X(\mathbb{C}) \rightarrow \mathrm{CH}_0(X)$$

$$\left(\sum_{i=1}^n x_i, \sum_{j=1}^n y_j \right) \mapsto \left[\sum_{i=1}^n x_i - \sum_{j=1}^n y_j \right].$$

For each integer $n \geq 1$, we have the morphism

$$\phi_n : E \times E' \rightarrow S^{2n}(E \times E') \times S^{2n}(E \times E')$$

$$(x, y) \mapsto (n(x, y) + n(0, 0), n(x, 0) + n(0, y)),$$

By [12, Theorem 1], $(\rho_{2n} \circ \phi_n)^{-1}(0)$ is a countable union of Zariski closed subsets of $E \times E'$.

On the other hand, since $p_g(E \times E') = 1$, the Albanese kernel $F^2\mathrm{CH}_0(E \times E')$ is “infinite dimensional” [9]; in particular, $F^2\mathrm{CH}_0(E \times E')_{\mathbb{Q}} \neq 0$. Since $F^2\mathrm{CH}_0(E \times E')$ is generated by cycles of the form $p(u) * p(v)$, it follows that $(\rho_{2n} \circ \phi_n)^{-1}(0)$ is a countable union of *proper* closed subsets of $E \times E'$. If D is a proper algebraic subset of $E \times E'$, then, by Lemma 2.2, $s^{-1}(D)$ is a proper analytic subset of $\mathbb{C} \setminus \{0, 1\}$, hence $s^{-1}(D)$ is countable. Thus, the set of $u \in \mathbb{C} \setminus \{0, 1\}$ such that $p(u) * p'(1 - u)$ is torsion is countable, which completes the proof in case both E and E' are non-singular.

If say $E' = E_0$, we use essentially the same proof. We let X be the open subscheme $E \times (E_0 \setminus \{*\})$ of $E \times E_0$. We have the map $\rho_n : S^n X(\mathbb{C}) \times S^n X(\mathbb{C}) \rightarrow \mathrm{CH}_0(E \times E_0)$ defined as above. By [6, Theorem 4.3], $(\rho_{2n} \circ \phi_n)^{-1}(0)$ is a countable union of closed subsets D_i of X . By [14], we have the similar infinite dimensionality result for $\mathrm{CH}_0(E \times E_0)$ as in the smooth case, from which it follows that each D_i is a proper closed subset of X . Thus, the closure of each D_i in $E \times E_0$ is a proper algebraic subset of $E \times E_0$. The same argument as in the smooth case finishes the proof. \square

3. INDETECTABILITY

The zero-cycle $p(u) * p(1 - u)$ is undetectable by cohomology theories built on the sheaf $\mathcal{O}_{E_{\mathrm{an}} \times E'_{\mathrm{an}}}^*$. We first consider the following abstract situation.

Let $\Gamma_0(2)$ be the complex:

$$\mathbb{Z}[\mathbb{C} \setminus \{0, 1\}] \rightarrow \mathbb{C}^* \otimes \mathbb{C}^*$$

$$u \mapsto u \otimes (1 - u),$$

with $\mathbb{C}^* \otimes \mathbb{C}^*$ in degree two.

Let $X = E \times E'$, and let $\Gamma(2)_{\text{an}}$ be a complex of sheaves on X_{an} with the following properties:

(3.1)

1. There is a group homomorphism $\text{cl} : \text{CH}_0(X) \rightarrow \mathbb{H}^4(X_{\text{an}}, \Gamma(2)_{\text{an}})$.
2. There is a map in the derived category of sheaves $D^b(\text{Sh}_{X_{\text{an}}})$, $\rho : \mathcal{O}_{X_{\text{an}}}^* \otimes \mathcal{O}_{X_{\text{an}}}^*[-2] \rightarrow \Gamma(2)_{\text{an}}$.
3. The composition

$$\mathbb{C}^* \otimes \mathbb{C}^*[-2] \rightarrow \mathcal{O}_{X_{\text{an}}}^* \otimes \mathcal{O}_{X_{\text{an}}}^*[-2] \rightarrow \Gamma(2)_{\text{an}}$$

extends to a map in $D^b(\text{Sh}_{X_{\text{an}}})$, $\Gamma_0(2) \rightarrow \Gamma(2)_{\text{an}}$.

4. The composition

$$\begin{aligned} \text{Pic}(X) \otimes \text{Pic}(X) &\cong H^1(X_{\text{an}}, \mathcal{O}_{X_{\text{an}}}^*) \otimes H^1(X_{\text{an}}, \mathcal{O}_{X_{\text{an}}}^*) \\ &\xrightarrow{\cup} H^2(X_{\text{an}}, \mathcal{O}_{X_{\text{an}}}^* \otimes \mathcal{O}_{X_{\text{an}}}^*) \xrightarrow{\rho} \mathbb{H}^4(X_{\text{an}}, \Gamma(2)_{\text{an}}) \end{aligned}$$

agrees with the composition

$$\text{Pic}(X) \otimes \text{Pic}(X) \xrightarrow{\cup} \text{CH}_0(X) \xrightarrow{\text{cl}} \mathbb{H}^4(X_{\text{an}}, \Gamma(2)_{\text{an}}).$$

Theorem 3.1. *Let $E = E_t$ and $E' = E_{t'}$, and let $\Gamma(2)_{\text{an}}$ be a complex of sheaves on $E_{\text{an}} \times E'_{\text{an}}$ satisfying the conditions (3.1). Then $\text{cl}(p(u) * p(1-u)) = 0$ for all $u \in \mathbb{C} \setminus \{0, 1\}$.*

Proof. We give the proof in case both E and E' are non-singular; the singular case is similar, but easier, and is left to the reader.

Since

$$p(u) * p(1-u) = [p_1^* c_1(L_u^{\text{alg}})] \cap [p_2^* c_1(L_{1-u}^{\text{alg}})],$$

it follows from (3.1)(4) that we need to show that $\rho([L_u^{\text{an}}] \cup [L_{1-u}^{\text{an}}]) = 0$. The class $[L_u^{\text{an}}] \in H^1(E_{\text{an}}, \mathcal{O}_{E_{\text{an}}}^*)$ is the image of $[L_u] \in H^1(E_{\text{an}}, \mathbb{C}^*)$ under the map of sheaves $\mathbb{C}^* \rightarrow \mathcal{O}_{E_{\text{an}}}^*$, and similarly for L_{1-u} and L_{1-u}^{an} . Thus, by (3.1)(3), it suffices to see that $p_1^*[L_u] \cup p_2^*[L_{1-u}] \in H^2(E \times E', \mathbb{C}^* \otimes \mathbb{C}^*)$ vanishes in $\mathbb{H}^4(E \times E', \Gamma_0(2))$.

The \mathbb{Z} -covers $p : \mathbb{C}^* \rightarrow E = E_t$, $p' : \mathbb{C}^* \rightarrow E' = E_{t'}$ give natural maps

$$\begin{aligned} \alpha : H^*(\mathbb{Z}, H^0(\mathbb{C}^*, \mathbb{C}^*)) &\rightarrow H^*(E_{\text{an}}, \mathbb{C}^*), \\ \beta : H^*(\mathbb{Z}, H^0(\mathbb{C}^*, \mathbb{C}^*)) &\rightarrow H^*(E'_{\text{an}}, \mathbb{C}^*). \end{aligned}$$

Similarly, the \mathbb{Z}^2 -cover $p \times p' : \mathbb{C}^* \times \mathbb{C}^* \rightarrow E \times E'$ gives the natural map

$$\gamma : \mathbb{H}^*(\mathbb{Z}^2, H^0(\mathbb{C}^* \times \mathbb{C}^*, \Gamma_0(2))) \rightarrow \mathbb{H}^*(E_{\text{an}} \times E'_{\text{an}}, \Gamma_0(2)).$$

Letting $\iota : \mathbb{C}^* \otimes \mathbb{C}^* \rightarrow \Gamma_0(2)$ denote the natural inclusion, the maps above are compatible with the respective cup products:

$$\iota \circ (\alpha(a) \cup \beta(b)) = \gamma \circ \iota(a \cup b).$$

Each $v \in \mathbb{C}^*$ gives the corresponding homomorphism $v : \mathbb{Z} \rightarrow \mathbb{C}^*$, $v(n) = v^n$. Since $[L_u] \in H^1(E_{\text{an}}, \mathbb{C}^*)$ is $\alpha(u : \mathbb{Z} \rightarrow \mathbb{C}^*)$ and $[L_{1-u}] \in H^1(E'_{\text{an}}, \mathbb{C}^*)$ is $\beta(1-u : \mathbb{Z} \rightarrow \mathbb{C}^*)$, it suffices to show that $\iota(p_1^*u \cup p_2^*(1-u)) = 0$ in $\mathbb{H}^4(\mathbb{Z}^2, \Gamma_0(2))$, where $p_1^*u, p_2^*(1-u) : \mathbb{Z}^2 \rightarrow \mathbb{C}^*$ are the respective homomorphisms $(a, b) \mapsto u^a$, and $(a, b) \mapsto (1-u)^b$.

We have the spectral sequence

$$E_2^{p,q} = H^p(\mathbb{Z}^2, H^q(\Gamma_0(2))) \implies \mathbb{H}^{p+q}(\mathbb{Z}^2, \Gamma_0(2)).$$

Since \mathbb{Z}^2 has cohomological dimension two, and since $H^q(\Gamma_0(2)) = 0$ for $q \neq 1, 2$, it follows that the natural map $\mathbb{H}^4(\mathbb{Z}^2, \Gamma_0(2)) \rightarrow H^2(\mathbb{Z}^2, H^2(\Gamma_0(2)))$ is an isomorphism. Since $H^2(\Gamma_0(2)) = K_2(\mathbb{C})$, we need to show that the image of $p_1^*u \cup p_2^*(1-u)$ in $H^2(\mathbb{Z}^2, K_2(\mathbb{C}))$ is zero.

By definition of the cup product in group cohomology, we have

$$\begin{aligned} [p_1^*u \cup p_2^*(1-u)]((a, b), (c, d)) &= p_1^*u(a, b) \otimes p_2^*(1-u)(c-a, d-b) \\ &= u^a \otimes (1-u)^{d-b}, \end{aligned}$$

which clearly vanishes in $K_2(\mathbb{C})$. □

Example 3.2. In [3], S. Bloch defines a quotient complex $\mathcal{B}(2)$ of the analytic complex $\mathcal{O}_{X_{\text{an}}}^*(1) \xrightarrow{2\pi i \otimes 1} \mathcal{O}_{X_{\text{an}}} \otimes \mathcal{O}_{X_{\text{an}}}^*$ fulfilling $\mathcal{H}^i(\mathcal{B}(2)) = 0$ for $i \neq 1, 2$,

$$\mathcal{H}^1(\mathcal{B}(2)) = \text{Im}\left(r : K_{3,\text{ind}}(\mathbb{C}) \rightarrow \mathbb{C}/\mathbb{Z}(2)\right) =: \Delta^*(1),$$

where r is the regulator map, and $\mathcal{H}^2(\mathcal{B}(2)) = \mathcal{K}_{2,\text{an}}$. He shows in the same article that $r(K_{3,\text{ind}}(\mathbb{C})) = r(K_{3,\text{ind}}(\bar{\mathbb{Q}}))$, thus $\Delta^*(1)$ is a countable subgroup of $\mathbb{C}/\mathbb{Z}(2)$, and also that $\mathcal{B}(2)$ maps to the complex $\mathbb{Z}(2) \rightarrow \mathcal{O}_{X_{\text{an}}} \rightarrow \Omega_{X_{\text{an}}}^1$ which computes the Deligne cohomology $H_{\mathcal{D}}^*(X, 2)$ when X is projective smooth over \mathbb{C} . In fact, the cycle map $\text{CH}^2(X) \rightarrow H_{\mathcal{D}}^4(X, 2)$ is shown to factor through $H_{\mathcal{D}}^4(X_{\text{an}}, \mathcal{B}(2))$ ([5]). S. Bloch ([4]) asked whether the cycle map $\text{CH}^2(X) \rightarrow H_{\mathcal{D}}^4(X_{\text{an}}, \mathcal{B}(2))$ could possibly be injective. The computations of this article show that it is not. Indeed, by Lemma (1.3) of [3], the complex $\Gamma_0(2)$ maps to the complex

$$\epsilon(\mathbb{Z}[\mathbb{C} \setminus \{0, 1\}]) \rightarrow \mathbb{C} \otimes \mathbb{C}^*,$$

where ϵ is defined via the dilogarithm function

$$\epsilon(a) = [\log(1-a) \otimes a] + [2\pi i \otimes \exp\left(\frac{-1}{2\pi i} \int_0^a \log(1-t) \frac{dt}{t}\right)],$$

and the latter complex maps to

$$\mathcal{B}(2)_X : \mathcal{O}_{X_{\text{an}}}^*(1) \xrightarrow{2\pi i \otimes 1} \mathcal{O}_{X_{\text{an}}} \otimes \mathcal{O}_{X_{\text{an}}}^* / (\epsilon(\mathbb{Z}[\mathbb{C} \setminus \{0, 1\}]))$$

for $X = \text{Spec } \mathbb{C}$. Let us take $\Gamma(2)_{\text{an}} = \mathcal{B}(2)$. We now verify the conditions 3.1. The condition 1 is given by [5]. Indeed, one computes the Leray spectral sequence associated to $\alpha : X_{\text{an}} \rightarrow X_{\text{Zar}}$ and the first term entering $H^4(\mathcal{B}(2))$ is

$$E^{2,2} = H_{\text{Zar}}^2(R\alpha_* \mathcal{B}(2)) = H^2(\mathcal{K}_{2,\mathbb{Z}}),$$

where $\mathcal{K}_{2,\mathbb{Z}} := \text{Ker}\left(\alpha_* \mathcal{K}_{2,\text{an}} \xrightarrow{d \log \wedge d \log} H^2(\mathbb{C}/\mathbb{Z}(2))\right)$. Then the cycle map cl is induced by $\mathcal{K}_2 \rightarrow \mathcal{K}_{2,\mathbb{Z}}$ on X_{Zar} , which is obviously compatible with the product in Pic . Thus we have 4. We have already discussed 2 and 3. Hence we can apply Theorem 2.3 to take a 0-cycle $p(u) * p(1-u)$ on $E \times E'$ where both E and E' are smooth elliptic curves which does not die in the Chow group $\text{CH}_0(E \times E')$, whereas it dies by Theorem 3.1 in $\mathbb{H}^4(\mathcal{B}(2))$.

In [8], S. Lichtenbaum constructs an étale version $\Gamma(2)$ of S. Bloch's analytic complex $\mathcal{B}(2)$, the cohomology of which contains $\text{CH}^2(X)$. This contrasts with the examples discussed above.

Over a p -adic field, W. Raskind and M. Spieß ([11]) show that the Albanese kernel modulo n of a product of two Tate elliptic curves is dominated by $K_2(k)/n$. This result is not immediately comparable to ours, but is obviously related.

4. THE RELATIVE SITUATION

In this section, we study the cycles constructed in section 2 on $X = E \times E_0$, where as there, E is smooth, and E_0 is a nodal curve. Let $\nu = 1 \times q : E \times \mathbb{P}^1 \rightarrow X$ be the normalization. We define

$$(4.1) \quad \bar{\mathcal{K}}_2 = \text{Ker}\left(\nu_* \mathcal{K}_2 \xrightarrow{|E \times 0| - |E \times \infty|} \mathcal{K}_2|_E\right)$$

Lemma 4.1. *One has*

$$\text{CH}^2(X) = H^2(X, \bar{\mathcal{K}}_2),$$

and the Chow group $\text{CH}_0(X)$ fits into an exact sequence

$$0 \rightarrow H^1(E, \mathcal{K}_2) \xrightarrow{\gamma} \text{CH}_0(X) \xrightarrow{\nu^*} \text{CH}_0(E \times \mathbb{P}^1) = \text{Pic}(E) \otimes \text{Pic}(\mathbb{P}^1) \rightarrow 0.$$

Moreover, the map γ is defined by

$$\gamma\left(\sum_{x \in E^{(1)}} x \otimes \lambda_x\right) = \sum_{x \in E^{(1)}} (x, p_0(\lambda_x)) - (x, 0).$$

Proof. The map $\nu^* : \mathcal{K}_2 \rightarrow \bar{\mathcal{K}}_2$ is obviously surjective, and by the Gersten resolution on the smooth points of X , the kernel is supported in codimension 1. Thus ν^* induces an isomorphism on H^2 .

On the other hand,

$$H^1(E \times \mathbb{P}^1, \mathcal{K}_2) = H^1(E, \mathcal{K}_2) \oplus H^0(E, \mathcal{K}_1) \cup c_1(\mathcal{O}(1)).$$

The term $H^1(E, \mathcal{K}_2)$ maps to $0 \in H^1(E, \mathcal{K}_2)$ via the difference of the restrictions to $E \times 0$ and $E \times \infty$, while $c_1(\mathcal{O}(1))$ restricts to 0 to either $E \times 0$ or $E \times \infty$. This shows the long exact sequence associated to the short one defining $\bar{\mathcal{K}}_2$.

Finally, the value $\gamma(x \otimes \lambda_x)$ of the map is given by the boundary morphism $\mathbb{C}^* \rightarrow H^1(X, \mathcal{O}_X^*)$ induced by the normalization sequence

$$0 \rightarrow \mathcal{O}_X^* \rightarrow q_* \mathcal{O}_{\mathbb{P}^1}^* \xrightarrow{|0|-\infty} \mathbb{C}^* \rightarrow 0$$

on the right argument λ_x . The formula for γ thus follows from Lemma 1.1. \square

Let $\text{Nm} : H^1(E, \mathcal{K}_2) \rightarrow \mathbb{C}^*$ be the norm map defined by

$$(4.2) \quad \text{Nm}\left(\sum_{x \in E^{(1)}} x \otimes \lambda_x\right) = \prod_{x \in E^{(1)}} \lambda_x.$$

We set

$$(4.3) \quad V(E) = \text{Ker Nm}.$$

One has

Lemma 4.2. $F^2\text{CH}_0(X) = \gamma(V(E)).$

Proof. By the definition given in §2, $F^2\text{CH}_0(X)$ is generated by the expressions $[(x, y)] - [(x, 0)] - [(0, y)] + [(0, 0)]$, with $x \in E(\mathbb{C})$ and $y \in E_0(\mathbb{C}) \setminus \{*\}$. By the formula for γ given in Lemma 4.1, this expression is $\gamma(x \otimes y - 0 \otimes y)$, after identifying $y \in \mathbb{C}^*$ with $p_0(y) \in E_0(\mathbb{C})$. Clearly $V(E)$ is generated by the elements of $H^1(E, \mathcal{K}_2)$ of the form $x \otimes y - 0 \otimes y$, whence the lemma. \square

Next we want to map $\text{CH}_0(X)$ to a relative version of S. Bloch's analytic motivic cohomology. So we define

$$(4.4) \quad \bar{\mathcal{B}}(2) = \text{Ker}\left(\nu_* \mathcal{B}(2) \xrightarrow{|E \times 0| - |E \times \infty|} \mathcal{B}(2)|_E\right)$$

In particular, $\bar{\mathcal{B}}(2)$ is an extension of

$$\bar{\mathcal{K}}_{2,\text{an}} = \text{Ker}\left(\nu_* \mathcal{K}_{2,\text{an}} \xrightarrow{|E \times 0^-|_{E \times \infty}} \mathcal{K}_{2,\text{an}}|_E\right)$$

placed in degree 2, by $\Delta^*(1)$, placed in degree 1. In other words, $\mathcal{B}(2)$ is the pull-back of $\bar{\mathcal{B}}(2)$ via the map $\nu^* : \mathcal{K}_{2,\text{an}} \rightarrow \bar{\mathcal{K}}_{2,\text{an}}$, and in particular, it receives the complex $\Gamma_0(2)$ as explained in the example 3.2.

Considering again the Leray spectral sequence attached to the identity $\alpha : X_{\text{an}} \rightarrow X_{\text{zar}}$, we see that

$$(4.5) \quad \bar{\mathcal{K}}_{2,\mathbb{Z}} := \text{Ker}\left(\alpha_* \bar{\mathcal{K}}_{2,\text{an}} \rightarrow \mathcal{H}^2(\mathbb{C}/\mathbb{Z}(2))\right)$$

receives $\bar{\mathcal{K}}_2$ and that the first map of the spectral sequence is then

$$(4.6) \quad H^2(X, \bar{\mathcal{K}}_{2,\mathbb{Z}}) \rightarrow \mathbb{H}^4(X_{\text{an}}, \bar{\mathcal{B}}(2)).$$

In conclusion, we have shown

Lemma 4.3. *One has a cycle map*

$$\psi_X : \text{CH}_0(X) \rightarrow \mathbb{H}^4(X_{\text{an}}, \bar{\mathcal{B}}(2))$$

compatible with the cycle map

$$\psi_{E \times \mathbb{P}^1} : \text{CH}_0(E \times \mathbb{P}^1) \rightarrow \mathbb{H}^4((E \times \mathbb{P}^1)_{\text{an}}, \mathcal{B}(2))$$

on the normalization. Moreover, ψ_X fulfills the conditions described in 3.1.

Proof. We just have to verify the condition 4 of 3.1. From the normalization sequence

$$0 \rightarrow \mathcal{O}_X^* \rightarrow \nu_* \mathcal{O}_{E \times \mathbb{P}^1}^* \xrightarrow{|E \times 0^-|_{E \times \infty}} \mathcal{O}_E^* \rightarrow 0,$$

one has a natural map

$$\mathcal{O}_{X_{\text{an}}}^* \otimes \mathcal{O}_{X_{\text{an}}}^* \rightarrow \bar{\mathcal{K}}_{2,\text{an}}$$

which obviously fulfills 3.1 4. □

Now we can apply Theorem 3.1 to conclude

Theorem 4.4. *The 0-cycles defined by the Steinberg curve on $E \times E_0$ die in the analytic motivic cohomology $\mathbb{H}^4(X_{\text{an}}, \bar{\mathcal{B}}(2))$.*

Let K be a subfield of \mathbb{C} . We next consider for any algebraic variety Z defined over K , the cycle map with values in the absolute Hodge cohomology

$$(4.7) \quad H^m(Z, \mathcal{K}_2) \xrightarrow{d \log \wedge d \log} H^m(Z, \Omega_{Z/\mathbb{Q}}^2)$$

induced by the absolute $d \log$ map

$$(4.8) \quad \mathcal{O}_Z^* \xrightarrow{d \log} \Omega_{Z/\mathbb{Q}}^1.$$

This cycle map is obviously compatible with the map γ , and with extension of scalars.

Let $E \rightarrow \operatorname{Spec} K$ be an elliptic curve over a subfield K of \mathbb{C} . We have the exact sheaf sequence

$$0 \rightarrow \mathcal{O}_E \otimes \Omega_{K/\mathbb{Q}}^1 \rightarrow \Omega_{E/\mathbb{Q}}^1 \rightarrow \Omega_{E/K}^1 \rightarrow 0,$$

which induces a two-term filtration $F^* \Omega_{E/\mathbb{Q}}^2$ of $\Omega_{E/\mathbb{Q}}^2$ with $F^2 \Omega_{E/\mathbb{Q}}^2 = \mathcal{O}_E \otimes \Omega_{K/\mathbb{Q}}^2$. This gives us the natural maps

$$\begin{aligned} \gamma_1 : H^*(E, \mathcal{O}_E) \otimes \Omega_{K/\mathbb{Q}}^1 &\rightarrow H^*(E, \Omega_{E/\mathbb{Q}}^1) \\ \gamma_2 : H^*(E, \mathcal{O}_E) \otimes \Omega_{K/\mathbb{Q}}^2 &\rightarrow H^*(E, \Omega_{E/\mathbb{Q}}^2). \end{aligned}$$

We have the norm map $\operatorname{Nm} : H^1(E, \mathcal{K}_2) \rightarrow H^0(K, \mathcal{K}_1) = K^*$ as in 4.2, but over K ; we let $V(E) \subset H^1(E, \mathcal{K}_2)$ be the kernel of Nm (see (4.3)).

Lemma 4.5. *Let K be an algebraically closed subfield of \mathbb{C} , $E \rightarrow \operatorname{Spec} K$ an elliptic curve over K . Then the cycle map with values in absolute Hodge cohomology maps $V(X)$ to the subgroup $\gamma_2[H^1(E, \mathcal{O}_E) \otimes \Omega_{E/\mathbb{Q}}^2]$ of $H^1(E, \Omega_{E/\mathbb{Q}}^2)$.*

Proof. The kernel of the composition

$$\operatorname{Pic}(E) = H^1(E, \mathcal{K}_1) \xrightarrow{d \log} H^1(E, \Omega_{E/\mathbb{Q}}^1) \rightarrow H^1(E, \Omega_{E/K}^1) \cong K$$

is the composition

$$\operatorname{Pic}(E) \xrightarrow{\deg} \mathbb{Z} \subset K,$$

hence the $d \log$ map sends $\operatorname{Pic}^0(E)$ to the subgroup $\gamma_1[H^1(E, \mathcal{O}_E) \otimes \Omega_{K/\mathbb{Q}}^1]$ of $H^1(E, \Omega_{E/\mathbb{Q}}^1)$.

Take $\tau \in \operatorname{Pic}^0(E)$, $u \in H^0(E, \mathcal{K}_1) = K^*$, and let $\xi = \tau \cup u \in H^1(E, \mathcal{K}_2)$. Then

$$d \log(\xi) = d \log(\tau) \cup d \log(u).$$

Since $d \log : K^* \rightarrow \Omega_{K/\mathbb{Q}}^1$ is just the absolute $d \log$ map, we see that $d \log(\xi)$ lands in the image of the cup product map

$$[H^1(E, \mathcal{O}_E) \otimes \Omega_{K/\mathbb{Q}}^1] \otimes \Omega_{K/\mathbb{Q}}^1 \rightarrow H^1(E, \Omega_{E/\mathbb{Q}}^2),$$

which is $\gamma_2(H^1(E, \mathcal{O}_E) \otimes \Omega_{K/\mathbb{Q}}^2)$.

Since K is algebraically closed, the cup product $\operatorname{Pic}(E) \otimes K^* \rightarrow H^1(E, \mathcal{K}_2)$ is surjective, from which one sees that the cup product maps

$\text{Pic}^0(E) \otimes K^*$ onto $V(E)$. Combining this with the computation above completes the proof. \square

From the surjectivity of the cup product $\text{Pic}^0(E) \otimes K^* \rightarrow V(E)$ for K algebraically closed, we see that the injection $H^1(E, \mathcal{K}_2) \rightarrow \text{CH}_0(X)$ sends $V(E)$ isomorphically onto $F^2\text{CH}_0(X)$.

Let K be a subfield of \mathbb{C} . We say that an element ξ of $\text{CH}_0(X)$ is *defined over K* if there is an K -scheme X^0 , an element ξ^0 of $\text{CH}_0(X^0)$ and an isomorphism $\alpha : X_{\mathbb{C}}^0 \rightarrow X$ such that $\xi = \alpha_*(\xi_{\mathbb{C}}^0)$. From Lemma 4.5 and the compatibility of $d\log$ with extension of scalars, we have

Lemma 4.6. *Take $K = \mathbb{C}$, and let ξ be an element of $F^2\text{CH}_0(X) = V(E)$. If ξ is defined over a field of transcendence degree one over \mathbb{Q} , then ξ vanishes under the cycle map to absolute Hodge cohomology.*

Corollary 4.7. *If E is an elliptic curve with complex multiplication, then there are non-torsion cycles $\xi \in F^2\text{CH}_0(X)$ dying in the analytic motivic cohomology as well as in absolute Hodge cohomology.*

Proof. By the remark above, we may replace $F^2\text{CH}_0(X)$ with $V(E)$. Let \bar{E} be a model for E , with equation $y^2 = 4x^3 - ax - b$ defined over a number field $K \subset \mathbb{C}$. Let $\omega = \frac{dx}{y}$ be the standard global one-form on \bar{E} .

Choosing an isomorphism $\bar{E}_{\mathbb{C}} \cong E_{\mathbb{C}}$ defines the period lattice $L_{\omega} \subset \mathbb{C}$ for ω . Choose a basis for L_{ω} of the form $\{\Omega, \tau\Omega\}$, and let $t = e^{2\pi i\tau}$. Let

$$\mathcal{P} : \mathbb{C} \rightarrow \mathbb{CP}^1$$

be the Weierstraß P -function for the lattice L_{ω} .

The map $\times\Omega^{-1} : \mathbb{C} \rightarrow \mathbb{C}$ gives rise to the isomorphism of Riemann surfaces $\alpha_{\text{an}} : \bar{E}_{\mathbb{C}}^{\text{an}} \rightarrow E_t^{\text{an}}$ making the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\times\Omega^{-1}} & \mathbb{C} \\ \downarrow (\mathcal{P}, \mathcal{P}') & & \downarrow \exp \\ & & \mathbb{C}^* \\ & & \downarrow p \\ \bar{E}_{\mathbb{C}}^{\text{an}} & \xrightarrow{\alpha_{\text{an}}} & E_t^{\text{an}} \end{array}$$

commute, i.e.,

$$p(u) = \alpha_{\text{an}}\left(\mathcal{P}\left(\frac{\Omega}{2\pi i}\log u\right), \mathcal{P}'\left(\frac{\Omega}{2\pi i}\log u\right)\right).$$

We let

$$\alpha : \bar{E}_{\mathbb{C}} \rightarrow E_t$$

be the corresponding isomorphism of algebraic elliptic curves over \mathbb{C} .

By [1], théorème 1, $\mathcal{P}(\frac{\Omega}{2\pi i}\log u)$ has transcendence degree 1 over $\bar{\mathbb{Q}}$ for all $u \in \mathbb{N}$, $u \geq 2$. (We thank Y. André for giving us this reference). Fix a $u \geq 2$, let K be the algebraic closure of the field $\mathbb{Q}(\mathcal{P}(\frac{\Omega}{2\pi i}\log u))$, and let $x \in \bar{E}(K)$ be the point $(\mathcal{P}(\frac{\Omega}{2\pi i}\log u), \mathcal{P}'(\frac{\Omega}{2\pi i}\log u))$. Then x is a generic point of \bar{E} over $\bar{\mathbb{Q}}$.

We take

$$\xi := p(u) * p(1 - u).$$

By construction, $\xi = \alpha(\xi_K \times_K \mathbb{C})$, where $\xi_K \in H^1(\bar{E}, \mathcal{K}_2)$ is the element $[(x) - (0)] \cup [1 - u]$. Here $[(x) - (0)]$ denotes the class in $\text{Pic}(E) = H^1(\bar{E}, \mathcal{K}_1)$, and $[1 - u]$ denotes the class in $H^0(\bar{E}, \mathcal{K}_1) = K^*$. Since K has transcendence degree one over $\bar{\mathbb{Q}}$, the class of ξ in the absolute Hodge cohomology of E vanishes, by Lemma 4.6. By Theorem 4.4, ξ dies in the analytic motivic cohomology of E as well. It remains to show that ξ is a non-torsion element of $H^1(E_K, \mathcal{K}_2)$.

We give an analytic proof of this using the regulator map with values in Deligne-Beilinson cohomology.

Let Y be a smooth projective surface over \mathbb{C} , and let $\text{NS}(Y)$ denote the Néron-Severi group of divisors modulo homological equivalence. Then Hodge theory implies that

$$\text{NS}(Y) = \{(z, \varphi) \in (H^2(Y_{\text{an}}, \mathbb{Z}(1)) \times F^1 H^2(Y_{\text{an}}, \mathbb{C})), z \otimes \mathbb{C} = \varphi\},$$

and that

$$\text{NS}(Y) \cap F^2 H_{DR}^2(Y) = \emptyset.$$

We note that the map $\text{Pic}(Y) \otimes \mathbb{C}^* \rightarrow H_{\mathcal{D}}^3(Y, \mathbb{Z}(2))$ induced by the cup product in Deligne cohomology factors through $\text{NS}(Y) \otimes \mathbb{C}^*$, and that the induced map $\iota : \text{NS}(Y) \otimes \mathbb{C}^* \rightarrow H_{\mathcal{D}}^3(Y, \mathbb{Z}(2))$ is injective. Indeed,

$$H_{\mathcal{D}}^3(Y, \mathbb{Z}(2)) = H^2(Y_{\text{an}}, \mathbb{C}/\mathbb{Z}(2))/F^2.$$

Now take $Y = E \times E$, and let $U \subset E$ be the complement of a non-empty finite set Σ of points of E . Let $[E \times 0]$ be the class of $E \times 0$ in $\text{NS}(Y)$, and let $\gamma : \mathbb{C}^* \rightarrow \text{NS}(Y) \otimes \mathbb{C}^*$ be the map $\gamma(v) = [E \times 0] \otimes v$. Let

$$\iota_U : \text{NS}(Y) \otimes \mathbb{C}^* \rightarrow H_{\mathcal{D}}^3(E \times U, \mathbb{Z}(2))$$

be the composition of ι with the restriction map $H_{\mathcal{D}}^3(Y, \mathbb{Z}(2)) \rightarrow H_{\mathcal{D}}^3(E \times U, \mathbb{Z}(2))$. We claim that the sequence

$$\mathbb{C}^* \xrightarrow{\gamma} \text{NS}(Y) \otimes \mathbb{C}^* \xrightarrow{\iota_U} H_{\mathcal{D}}^3(E \times U, \mathbb{Z}(2))$$

is exact. Indeed, we have the localization sequence

$$\bigoplus_{s \in \Sigma} H_{\mathcal{D}}^1(E \times s, \mathbb{Z}(1)) \xrightarrow{\oplus s \iota_s} H_{\mathcal{D}}^3(Y, \mathbb{Z}(2)) \rightarrow H_{\mathcal{D}}^3(E \times U, \mathbb{Z}(2)) \rightarrow,$$

the isomorphism $H_{\mathcal{D}}^1(E \times s, \mathbb{Z}(1)) \cong \mathbb{C}^*$ and the identity

$$\iota_s(v) = \gamma(v), \quad v \in \mathbb{C}^*,$$

which proves our claim.

In particular, let $[\Xi] = [\Delta - \{0\} \times E] \otimes v$, where Δ is the diagonal, v is an element of \mathbb{C}^* which is not a root of unity, and $[\Delta - \{0\} \times E]$ is the class in $\text{NS}(Y)$. Since $[\Delta - \{0\} \times E]$ is not torsion in $\text{NS}(Y)/[E \times \{0\}]$, we see that $[\Xi]$ has non-torsion image $[\Xi_{\mathbb{C}(E)}]$ in

$$H_{\mathcal{D}}^3(E \times_{\mathbb{C}} \mathbb{C}(E), \mathbb{Z}(2)) := \lim_{\emptyset \neq U \subset E} H_{\mathcal{D}}^3(E \times U, \mathbb{Z}(2)),$$

where the limit is over non-empty Zariski open subsets U of E .

Let Ξ be the image of $(\Delta - 0 \times E) \otimes v$ in $H^1(Y, \mathcal{K}_2)$. Then $[\Xi]$ is the image of Ξ under the regulator map $H^1(Y, \mathcal{K}_2) \rightarrow H_{\mathcal{D}}^3(Y, \mathbb{Z}(2))$. Similarly, letting $\Xi_{\mathbb{C}(E)}$ be the pull-back of Ξ to $E \times_{\mathbb{C}} \mathbb{C}(E)$, $[\Xi_{\mathbb{C}(E)}]$ is the image of $\Xi_{\mathbb{C}(E)}$ under the regulator map $H^1(E \times_{\mathbb{C}} \mathbb{C}(E), \mathcal{K}_2) \rightarrow H_{\mathcal{D}}^3(E \times_{\mathbb{C}} \mathbb{C}(E), \mathbb{Z}(2))$. Thus, $\Xi_{\mathbb{C}(E)}$ is a non-torsion element of $H^1(E \times_{\mathbb{C}} \mathbb{C}(E), \mathcal{K}_2)$ for each non-torsion element $v \in \mathbb{C}^*$.

Let $\bar{\Delta}$ be the diagonal in $\bar{E} \times \bar{E}$, let $\bar{\xi}$ be the image of $(\bar{\Delta} - 0 \times \bar{E}) \otimes (1 - u)$ in $H^1(\bar{E}, \mathcal{K}_2)$, and let $\bar{\xi}_{\bar{\mathbb{Q}}(E)}$ be the image of $\bar{\xi}$ in $H^1(\bar{E} \times_{\bar{\mathbb{Q}}} \bar{\mathbb{Q}}(\bar{E}), \mathcal{K}_2)$. Clearly, after choosing a complex embedding $\bar{\mathbb{Q}} \subset \mathbb{C}$, $\Xi_{\mathbb{C}(E)}$ (for $v = 1 - u$) is the image of $\bar{\xi}_{\bar{\mathbb{Q}}(E)}$ under the extension of scalars $\bar{\mathbb{Q}}(\bar{E}) \rightarrow \mathbb{C}(\bar{E}) \cong \mathbb{C}(E)$, hence $\bar{\xi}_{\bar{\mathbb{Q}}(E)}$ is a non-torsion element of $H^1(\bar{E} \times_{\bar{\mathbb{Q}}} \bar{\mathbb{Q}}(\bar{E}), \mathcal{K}_2)$.

Since x is a geometric generic point of \bar{E} over $\bar{\mathbb{Q}}$, there is an embedding $\sigma : \bar{\mathbb{Q}}(E) \rightarrow \mathbb{C}$ such that $x : \text{Spec } \mathbb{C} \rightarrow \bar{E}$ is the composition $\text{Spec } \mathbb{C} \rightarrow \text{Spec } \bar{\mathbb{Q}}(E) \rightarrow \bar{E}$. Thus, ξ is the image of $\bar{\xi}$ under $(\text{id} \times x)^* : H^1(\bar{E} \times_{\bar{\mathbb{Q}}} \bar{E}, \mathcal{K}_2) \rightarrow H^1(E, \mathcal{K}_2)$, and hence ξ is the image of $\bar{\xi}_{\bar{\mathbb{Q}}(E)}$ under the map $\text{id} \times \sigma_* : H^1(\bar{E} \times_{\bar{\mathbb{Q}}} \bar{\mathbb{Q}}(\bar{E}), \mathcal{K}_2) \rightarrow H^1(E, \mathcal{K}_2)$ induced by the extension of scalars σ .

Since the kernel of $\text{id} \times \sigma_*$ is torsion, it follows that ξ is a non-torsion element of $H^1(E, \mathcal{K}_2)$, as desired. \square

Remark 4.8. Going back to $X = E \times E'$, where both elliptic curves are smooth, we are lacking the transcendence theorem which would force the existence of a cycle $0 \neq \xi = p(u) * p(1 - u) \in F^2\text{CH}_0(X)$ dying both in $\mathbb{H}^4(X, \mathcal{B}(2))$ and in absolute Hodge cohomology.

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